

CANONICAL STABILITY IN TERMS OF SINGULARITY INDEX FOR ALGEBRAIC THREEFOLDS

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(Received 4 February 2000; revised 19 April 2000)¹

Introduction

Throughout the ground field is always supposed to be algebraically closed of characteristic zero. Let X be a smooth projective threefold of general type, denote by ϕ_m the m -canonical map of X which is nothing but the rational map naturally associated with the complete linear system $|mK_X|$. Since, once given such a 3-fold X , ϕ_m is birational whenever $m \gg 0$, thus a quite interesting thing is to find the optimal bound for such an m . This bound is important because it is not only crucial to the classification theory, but also strongly related to other problems. For example, it can be applied to determine the order of the birational automorphism group of X ([21], Remark in §1). To fix the terminology, we say that ϕ_m is *stably birational* if ϕ_t is birational onto its image for all $t \geq m$. It is well-known that the parallel problem in surface case was solved by Bombieri ([1]) and others. In the 3-dimensional case, many authors have studied the problem, in quite different ways. Because, in this paper, we are interested in the results obtained by M. Hanamura ([7]), we don't plan to mention more references here. According to 3-dimensional MMP, X has a minimal model which is a normal projective 3-fold with only \mathbb{Q} -factorial terminal singularities. Though X may have many minimal models, the singularity index (namely the canonical index) of any of its minimal models is uniquely determined by X . Denote by r the canonical index of minimal models of X . When $r = 1$, we know that ϕ_6 is stably birational by virtue of [3], [6], [13] and [14]. When $r \geq 2$, M. Hanamura proved the following theorem.

THEOREM 0. (*Theorem (3.4) of [7]*) *Let X be a smooth projective threefold of general type with a minimal model of the canonical index r . Then $\phi_{n_0(r)}$ is stably birational onto its image, where $n_0(r)$ is a function defined as*

$$\begin{array}{cccc} * & r = 2 & 3 \leq r \leq 5 & r \geq 6 \\ n_0(r) & 13 & 4r + 4 & 4r + 3. \end{array}$$

* The author was partially supported by both the Post-Doc Fellowship of the Mathematisches Institut der Universität Göttingen, Germany and the National Natural Science Foundation of China.

¹Accepted for publication in **Math. Proc. Camb. Phil. Soc.**

Noting that the output $n_0(1)$ of Hanamura's method is actually 7 (rather than 6), it is reasonable to believe that the bound in Theorem 0 is not optimal. On the other hand, we don't know whether the canonical index r is bounded or not, actually r can be strangely large for some X . This suggests that to find the optimal bounds for $n_0(r)$ should still be very interesting. As far as our method can tell in this paper, the results are as the following

MAIN THEOREM. *Let X be a smooth projective threefold of general type with a minimal model of the canonical index r . Then*

(i) ϕ_m is generically finite whenever $m \geq l_0(r)$, where $l_0(r)$ is a function defined as

$$\begin{array}{cccccc} * & r = 2 & 3 \leq r \leq 5 & r \geq 6 \\ l_0(r) & 10 & 2r + 5 & 2r + 4. \end{array}$$

(ii) $\phi_{m_0(r)}$ is stably birational onto its image, where $m_0(r)$ is a function defined as

$$\begin{array}{ccccccc} * & r = 2 & r = 3 & r = 4 & r = 5 & r = 6 & r \geq 7 \\ m_0(r) & 11 & 15 & 17 & 18 & 19 & 2r + 6. \end{array}$$

As an application of our method, we shall present the following

COROLLARY. *Let X be a smooth projective 3-fold of general type. Then ϕ_9 is birational if $p_g(X) \geq 2$.*

Remark. The above corollary is an improvement to Kollar's result (Corollary 4.8 of [11]) that ϕ_{16} is birational if $p_g(X) \geq 2$. Actually, Kollar proved there that ϕ_{11k+5} is birational if $P_k := h^0(X, kK_X) \geq 2$, where k is a positive integer. Recently, [4] improved this result to the level that either ϕ_{7k+3} or ϕ_{7k+5} is birational under the same condition.

For readers' convenience, we briefly explain the whole technique of this paper. According to Hanamura's result that $|(r+2)K_X|$ is not composed of a pencil, we can take a general member S_2 of the movable part of this system. Actually we can suppose that S_2 is smooth. Then we use the Matsuki-Tankeev principle to reduce the birationality problem to a parallel one for the adjoint system $|K + L|$ on the surface S_2 which is a smooth projective surface of general type. We shall inevitably treat a very delicate case in which L is the round-up of certain nef and big \mathbb{Q} -divisor A , i.e. $L = \lceil A \rceil$. Instead of applying Reider's result, we go on reducing to the problem on a curve. The technical point is to estimate the degree of the divisor in question on the curve. The Kawamata-Ramanujam-Viehweg vanishing theorem played an important role in the whole context.

1. Preliminaries

Let X be a normal projective variety of dimension d . We denote by $\text{Div}(X)$ the group of Weil divisors on X . An element $D \in \text{Div}(X) \otimes \mathbb{Q}$ is called a \mathbb{Q} -divisor. A \mathbb{Q} -divisor D is said to be \mathbb{Q} -Cartier if mD is a Cartier divisor for some positive integer m . For a \mathbb{Q} -Cartier divisor D and an irreducible curve $C \subset X$, we can define the intersection number $D \cdot C$ in a natural way. A \mathbb{Q} -Cartier divisor D is called

nef (or *numerically effective*) if $D \cdot C \geq 0$ for any effective curve $C \subset X$. A nef divisor D is called *big* if $D^d > 0$. We say that X is \mathbb{Q} -*factorial* if every Weil divisor on X is \mathbb{Q} -Cartier. For a Weil divisor D on X , write $\mathcal{O}_X(D)$ as the corresponding reflexive sheaf. Denote by K_X a canonical divisor of X , which is a Weil divisor. X is called *minimal* if K_X is a nef \mathbb{Q} -Cartier divisor. X is said to be of general type if the Kodaira dimension $\text{kod}(X) = \dim(X)$. For a positive integer m , we set $\omega_X^{[m]} := \mathcal{O}_X(mK_X)$ and call $P_m(X) := \dim_k H^0(X, \omega_X^{[m]})$ the m -th plurigenus of X . We remark that $P_m(X)$ is an important birational invariant.

X is said to have only *canonical singularities* (resp. *terminal singularities*) according to Reid ([15]) if the following two conditions hold:

- (i) for some positive integer r , rK_X is Cartier;
- (ii) for some resolution $f : Y \rightarrow X$, $K_Y = f^*(K_X) + \sum a_i E_i$ for $0 \leq a_i \in \mathbb{Q}$ (resp. $0 < a_i \forall i$, where the E_i vary amongst all the exceptional divisors on Y). The minimal r that satisfies (i) is called *the canonical index* of X and can be also written as $r(X)$.

According to Mori's MMP ([10], [12]), when V is a smooth projective threefold, there exists a birational map $\sigma : V \dashrightarrow X$ where X can be a minimal 3-fold with only \mathbb{Q} -factorial terminal singularities. Usually, X is not uniquely determined by V , but the canonical index $r(X)$ is.

Let $D = \sum a_i D_i$ be a \mathbb{Q} -divisor on X where the D_i are distinct prime divisors and $a_i \in \mathbb{Q}$. We define

$$\begin{aligned} \text{the round-down } \lfloor D \rfloor &:= \sum \lfloor a_i \rfloor D_i, \text{ where } \lfloor a_i \rfloor \text{ is the integral part of } a_i, \\ \text{the round-up } \lceil D \rceil &:= -\lfloor -D \rfloor, \\ \text{the fractional part } \{D\} &:= D - \lfloor D \rfloor. \end{aligned}$$

Remark 1.1. Suppose that X has only canonical singularities and that $f : V \rightarrow X$ is a resolution. We have

$$P_m(X) = h^0\left(V, \mathcal{O}_V(\lfloor f^*(mK_X) \rfloor)\right) = h^0\left(V, \mathcal{O}_V(\lceil f^*(mK_X) \rceil)\right) = P_m(V)$$

for any positive integer m .

We always use the Kawamata-Ramanujam-Viehweg vanishing theorem in the following form.

VANISHING THEOREM. ([9] or [18]) *Let X be a smooth complete variety, $D \in \text{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions:*

- (i) D is nef and big;
- (ii) the fractional part of D has supports with only normal crossings. Then $H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for all $i > 0$.

Most of our notations are standard within algebraic geometry except the following which we are in favor of: \sim_{lin} means *linear equivalence* while \sim_{num} means *numerical equivalence*.

2. Some lemmas

2.1 The Matsuki-Tankeev principle. This principle is tacitly used throughout our argument. Suppose X is a smooth variety, $|M|$ is a base point free system on X and D is a divisor with $|D| \neq \emptyset$. We want to know when $\Phi_{|D+M|}$ is birational. The following principles are due to Tankeev and Matsuki respectively.

(P1). (*Lemma 2 of [17]*) Suppose $|M|$ is not composed of a pencil, i.e.

$$\dim\Phi_{|M|}(X) \geq 2$$

and take a general member $Y \in |M|$. If the restriction of $\Phi_{|D+M|}$ to Y is birational, then $\Phi_{|D+M|}$ is birational.

(P2). (*see the proof of the main theorem in [14]*) Suppose $|M|$ is composed of a pencil and take the Stein factorization of

$$\Phi_{|M|} : X \xrightarrow{f} C \longrightarrow W \subset \mathbb{P}^N,$$

where W is the image of X through $\Phi_{|M|}$ and f is a fibration onto a smooth curve C . Let F be a general fiber of f . If we know (say by the vanishing theorem) that $\Phi_{|D+M|}$ can distinguish general fibers of f (i.e. separates any two points of the respective fibers) and its restriction to F is birational, then $\Phi_{|D+M|}$ is also birational.

LEMMA 2.2. Let X be a smooth projective variety of dimension d , $D \in \text{Div}(X) \otimes \mathbb{Q}$ be a \mathbb{Q} -divisor on X . Then the following assertions are true:

- (i) if S is a smooth reduced irreducible divisor on X and S is not a fractional component of D , then $\lceil D \rceil|_S \geq \lceil D|_S \rceil$;
- (ii) if $\pi : X' \longrightarrow X$ is a birational morphism, then $\pi^*(\lceil D \rceil) \geq \lceil \pi^*(D) \rceil$.

Proof. This lemma is very easy to check. \square

LEMMA 2.3. Let S be a smooth projective surface of general type and L be a nef and big divisor on S . Then $\Phi_{|K_S + mL|}$ is birational in the following cases:

- (i) $m \geq 4$;
- (ii) $m = 3$ and $L^2 \geq 2$.

Proof. This is a direct result of Corollary 2 of [16]. \square

LEMMA 2.4. (*Lemma (3.2) of [7]*) Let X be a minimal threefold of general type with canonical index $r \geq 2$. Then $\dim\phi_{mr+s}(X) \geq 2$ in the following cases, where m is a positive integer and $0 \leq s < r$:

- (i) $r = 2$ and $m \geq 3$;
- (ii) $r = 3$ and $m \geq 2$;
- (iii) $r = 4, 5$, $0 \leq s \leq 2$ and $m \geq 2$; $r = 4, 5$, $s \geq 3$ and $m \geq 1$;
- (iv) $r \geq 6$, $0 \leq s \leq 1$ and $m \geq 2$; $r \geq 6$, $s \geq 2$ and $m \geq 1$.

LEMMA 2.5. Under the same assumption as in Lemma 2.4, the plurigenus $P_{mr+s}(X) \geq 3$ in one of the following cases:

- (i) $r = 2$ and $m \geq 2$;
- (ii) $r \geq 3$, $0 \leq s \leq 1$ and $m \geq 2$; $r \geq 3$, $s \geq 2$ and $m \geq 1$.

Proof. This is obvious from the proof of Lemma (3.2) in [7]. In order to be precise and to cite it many times in this paper, let us recall the estimation there.

When $r \geq 3$, r is even and $s \geq 2$, we have

$$\begin{aligned} P_{mr+s}(X) &\geq \frac{1}{12} \left\{ 2r^2m^3 + (6s - 3)rm^2 + \left(6s^2 - 6s - \frac{1}{2}r^2 \right)m \right\} (rK_X^3) \\ &\geq \frac{1}{12} (r^2 + 6r + 9) \end{aligned} \tag{2.1}$$

When $r \geq 3$, r is odd and $s \geq 2$, we have

$$\begin{aligned} P_{mr+s}(X) &\geq \frac{1}{12} \left\{ (mr+s)(mr+s-1)(2mr+2s-1) + m \left(-\frac{1}{2}r^3 + \frac{1}{2}r \right) \right. \\ &\quad \left. - s(s-1)(2s-1) \right\} (K_X^3) \\ &\geq \frac{1}{8}(r^2 + 6r + 8) \end{aligned} \tag{2.2}$$

When $s = 1$, we have

$$P_{mr+1}(X) \geq \frac{1}{12}r(m^2 - 1)(2rm + 3)(rK_X^3) \tag{2.3}$$

When $s = 0$, we have

$$P_{mr}(X) \geq \frac{1}{12}r(m^2 - 1)(2rm - 3)(rK_X^3) \tag{2.4}$$

The above four formulae give the result. \square

The following lemma is sufficient to derive our result, though it seems that one might exploit its potential.

LEMMA 2.6. *Let S be a smooth projective surface of general type and L be a nef divisor on S such that $|L|$ gives a generically finite map. Then*

- (i) $L^2 \geq h^0(S, L) - 2$; if $\Phi_{|L|}$ is not birational, then $L^2 \geq 2h^0(S, L) - 4$.
- (ii) if $p_g(S) > 0$, then $L^2 \geq 2h^0(S, L) - 4$.
- (iii) $K_S + L$ is always effective.

Proof. The first part is trivial. One should note that a non-degenerate surface in \mathbb{P}^N has degree $\geq N - 1$. In order to prove the second part, we may suppose that $|L|$ is base point free. Let C be a general member of $|L|$, then

$$h^0(C, L|_C) \geq h^0(S, L) - 1.$$

Noting that C is moving and that we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_S(K_S - C) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow \mathcal{O}_C(K_S|_C) \longrightarrow 0,$$

the inclusion

$$H^0(S, K_S - C) \hookrightarrow H^0(S, K_S)$$

is proper. So $h^0(C, K_S|_C) > 0$, which means $h^1(C, L|_C) > 0$ and $L|_C$ is special. Thus Clifford's theorem implies that

$$L^2 = \deg(L|_C) \geq 2h^0(C, L|_C) - 2 \geq 2h^0(S, L) - 4.$$

Finally, (iii) is an easy exercise by Riemann-Roch. \square

LEMMA 2.7. *Let X be a smooth projective variety of dimension ≥ 2 . Let D be a divisor on X , $h^0(X, \mathcal{O}_X(D)) \geq 2$ and S be a smooth irreducible divisor on X such that S is not a fixed component of $|D|$. Denote by M the movable part of $|D|$ and by N the movable part of $|D|_S|$ on S . Suppose the natural restriction map*

$$H^0(X, \mathcal{O}_X(D)) \xrightarrow{\theta} H^0(S, \mathcal{O}_S(D|_S))$$

is surjective. Then $M|_S \geq N$ and thus

$$h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(N)) = h^0(S, \mathcal{O}_S(D|_S)).$$

Proof. Denote by Z the fixed part of $|D|$. Because S is not a fixed component of $|Z|$, we see that $Z|_S \geq 0$. Thus $D|_S \geq M|_S$. Considering the natural map

$$H^0(X, \mathcal{O}_X(M)) \xrightarrow{\theta_0} H^0(S, \mathcal{O}_S(M|_S)),$$

we have

$$h^0(S, \mathcal{O}_S(M|_S)) \geq \dim_{\mathbb{C}} \text{im}(\theta_0) = \dim_{\mathbb{C}} \text{im}(\theta) = h^0(S, \mathcal{O}_S(D|_S)).$$

This means $h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(D|_S))$ and so $M|_S \geq N$ on S . \square

3. The generic finiteness

This section is devoted to study the generic finiteness of ϕ_m . Whenever we mention a minimal 3-fold, we mean one with only \mathbb{Q} -factorial terminal singularities. The following theorem is the easy part of the Main Theorem.

THEOREM 3.1. *Let X be a projective minimal 3-fold of general type with $2 \leq r(X) \leq 5$. Then ϕ_{4r+3} is stably birational.*

Proof. According to Lemma 2.5, $P_{m_1}(X) \geq 3$ for $m_1 \geq r + 2$. Take necessary blowing-ups $\pi : X' \longrightarrow X$ along nonsingular centers, according to Hironaka, such that X' is smooth and $|m_1 K_{X'}|$ defines a morphism (of course, $|m_1 K_{X'}|$ may have fixed components). Denote by M the movable part of $|m_1 K_{X'}|$. We have

$$|K_{X'} + 3\pi^*(rK_X) + M| \subset |(m_1 + 3r + 1)K_{X'}|.$$

First we note that $K_{X'} + 3\pi^*(rK_X)$ is effective according to Lemma 2.5. If $|M|$ is not composed of a pencil, then a general member of it is an irreducible smooth projective surface S of general type. Set $L := \pi^*(rK_X)|_S$, which is a nef and big Cartier divisor on S with $L^2 \geq 2$. Using the vanishing theorem, we get

$$|K_{X'} + 3\pi^*(rK_X) + S| |_S = |K_S + 3L|.$$

The right system gives a birational map by Lemma 2.3. So (P1) implies what we want in this case. If $|M|$ is composed of a pencil, we take the Stein-factorization of

$$\Phi : X' \dashrightarrow Y' \xrightarrow{f} C \dashrightarrow W$$

where W is the image of X' through $\Phi_{|M|}$ and f is a fibration onto the smooth curve C . Generically, M can be written as a disjoint union of fibers of f , i.e. $M \sim_{\text{lin}} \sum_{i=1}^a F_i$. The F_i 's are irreducible smooth surfaces of general type. The effectiveness of $K_{X'} + 3\pi^*(rK_X) + M$ implies that $|K_{X'} + 3\pi^*(rK_X) + M|$ can distinguish general fibers of the morphism $\Phi_{|M|}$. On the other hand, we have the following exact sequence

$$H^0(X', K_{X'} + 3\pi^*(rK_X) + M) \longrightarrow \bigoplus_{i=1}^a H^0(F_i, K_{F_i} + 3L_i) \longrightarrow 0$$

where $L_i := \pi^*(rK_X)|_{F_i}$ is a nef and big divisor with $L_i^2 \geq 2$. This shows that the system $|K_{X'} + 3\pi^*(rK_X) + M|$ can also distinguish different components in a general fiber of $\Phi_{|M|}$ and the restriction to each F_i gives a birational map. Thus, by (P2), we have completed the proof. \square

Remark 3.2. Throughout this paper, we shall deal with the same situation as in the proof of Theorem 3.1. In order to avoid unnecessary redundancy, we give the definition of so-called *generic irreducible element* of a moving system $|M|$ on a variety V . Using our notations in the proof of Theorem 3.1, we shall call S (respectively, F_i) a generic irreducible element of $|M|$ ignoring whether it is composed of a pencil or not. In our case, we always use both (P1) and (P2).

THEOREM 3.3. *Let X be a projective minimal threefold of general type with the canonical index r . Then ϕ_m is generically finite whenever $m \geq l_0(r)$, where $l_0(r)$ is a function defined as the following*

$$l_0(r) = \begin{cases} 10 & \text{if } r = 2, \\ 2r + 5 & \text{if } 3 \leq r \leq 5, \\ 2r + 4 & \text{if } r \geq 6. \end{cases}$$

Proof. The idea of the proof is quite simple. We formulate our proof through steps.

Step 1. Set up for the proof.

First, we define

$$m_2 = \begin{cases} 6 & \text{if } r = 2, \\ r + 3 & \text{if } 3 \leq r \leq 5, \\ r + 2 & \text{if } r \geq 6. \end{cases}$$

Take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that

- (1) X' is smooth;
- (2) $|m_2 K_{X'}|$ defines a morphism;
- (3) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by M'_2 the movable part of $|m_2 K_{X'}|$ and by S'_2 a general member of $|M'_2|$. From Lemma 2.4, we know that $\dim \phi_{m_2}(X) \geq 2$. Given any integer $t \geq r + 2$, we know that $|tK_{X'}|$ is always effective according to Lemma 2.5. If $|M'_2|$ has already given a generically finite map, then ϕ_{t+m_2} is generically finite and thus the theorem is true in this situation. So from now on, we suppose $\dim \phi_{m_2}(X) = 2$. From (2.1), (2.2), (2.3) and (2.4), we have

$$h^0(X', S'_2) = P_{m_2}(X) \geq \begin{cases} 12 & \text{if } r = 2, \\ 7 & \text{if } r = 3, \\ \frac{1}{8}(r^2 + 10r + 24) & \text{if } 4 \leq r \leq 5, \\ \frac{1}{8}(r^2 + 6r + 24) & \text{if } r \geq 6. \end{cases}$$

On the surface S'_2 , we set $L'_2 := M'_2|_{S'_2}$. Then $|L'_2|$ is composed of a pencil. We can write

$$L'_2 \sim_{\text{lin}} \sum_{i=1}^{a'_2} C_i \sim_{\text{num}} a'_2 C,$$

where $a'_2 \geq h^0(S'_2, L'_2) - 1$ and C is a generic irreducible element of $|L'_2|$. Since $h^0(S'_2, L'_2) \geq h^0(X', S'_2) - 1$, we get

$$a'_2 \geq \begin{cases} 10 & \text{if } r = 2, \\ 5 & \text{if } r = 3, \\ \frac{1}{8}(r^2 + 10r + 8) & \text{if } 4 \leq r \leq 5, \\ \frac{1}{8}(r^2 + 6r - 8) & \text{if } r \geq 6. \end{cases}$$

Step 2. Reduce to the problem on a curve.

Because

$$|K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp + S'_2| \subset |(t+m_2)K_{X'}|,$$

it is sufficient to prove the generic finiteness of $\Phi_{|K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp + S'_2|}$. Noting that $K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp$ is effective, we only have to verify that

$$|K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp + S'_2| \mid_{S'_2}$$

gives a generically finite map by virtue of (P1).

The vanishing theorem gives

$$|K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp + S'_2| \mid_{S'_2} = |K_{S'_2} + \Gamma(t-1)\pi^*(K_X)^\perp|_{S'_2},$$

so we are reduced to verify the same property for

$$|K_{S'_2} + \Gamma(t-1)\pi^*(K_X)^\perp|_{S'_2}|.$$

Since

$$K_{S'_2} + \Gamma(t-1)\pi^*(K_X)^\perp|_{S'_2} = \left(K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp \right) \mid_{S'_2} + L'_2$$

and $K_{X'} + \Gamma(t-1)\pi^*(K_X)^\perp$ is effective by Lemma 2.5, the system

$$|K_{S'_2} + \Gamma(t-1)\pi^*(K_X)^\perp|_{S'_2}|$$

can distinguish general fibers of $\Phi_{|L'_2|}$. So it is sufficient to show that

$$\Phi_{|K_{S'_2} + \Gamma(t-1)\pi^*(K_X)^\perp|_{S'_2}|} \mid_C$$

is a finite map for a generic irreducible element C of $|L'_2|$.

Step 3. Verifying the finiteness on C .

Since $m_2\pi^*(K_X) \geq S'_2$, we can write

$$\pi^*(K_X)^\perp \mid_{S'_2} = L'_2 + F \quad \text{and} \quad a'_2 C + F$$

where $E_{\mathbb{Q}}$ is an effective \mathbb{Q} -divisor on S'_2 . So we have

$$(t-1)\pi^*(K_X)|_{S'_2} \sim_{\text{num}} \frac{(t-1)a'_2}{m_2}C + \frac{t-1}{m_2}E_{\mathbb{Q}}$$

and

$$(t-1)\pi^*(K_X)|_{S'_2} - C - \frac{1}{a'_2}E_{\mathbb{Q}} \sim_{\text{num}} (t-1)\left(1 - \frac{m_2}{(t-1)a'_2}\right)\pi^*(K_X)|_{S'_2}.$$

Set $\alpha := 1 - \frac{m_2}{(t-1)a'_2}$, it is easy to verify that $\alpha > 0$. This shows that

$$H^1\left(S'_2, K_{S'_2} + {}^\lceil(t-1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_{\mathbb{Q}}\rceil - C\right) = 0$$

according to the vanishing theorem. Thus we have the exact sequence

$$H^0\left(S'_2, K_{S'_2} + {}^\lceil(t-1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_{\mathbb{Q}}\rceil\right) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := {}^\lceil(t-1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_{\mathbb{Q}}\rceil|_C$ is a divisor on C with positive degree. In fact,

$$\deg(D) \geq (t-1)\alpha\pi^*(K_X)|_{S'_2} \cdot C > 0.$$

Noting that C is a smooth curve of genus ≥ 2 , we see that $|K_C + D|$ gives a finite map. Therefore

$$\Phi|_{K_{S'_2} + {}^\lceil(t-1)\pi^*(K_X)|_{S'_2} - \frac{1}{a'_2}E_{\mathbb{Q}}\rceil}|_C$$

is generically finite. So $\Phi|_{K_{S'_2} + {}^\lceil(t-1)\pi^*(K_X)|_{S'_2}}|_C$ is also generically finite. This derives the generic finiteness of ϕ_{t+m_2} . \square

4. The birationality: $r \geq 4$

Suppose X is a projective minimal 3-fold of general type with the canonical index r . First we take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that

- (i) X' is smooth;
- (ii) the system $|(r+2)K_{X'}|$ defines a morphism;
- (iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by M_2 the movable part of $|(r+2)K_{X'}|$. We know from Lemma 2.4 that $|M_2|$ is not composed of a pencil when $r \geq 6$.

From now on, we assume that $r \geq 4$ and that $|(r+2)K_{X'}|$ is not composed of a pencil. Take a general member $S_2 \in |M_2|$. Then S_2 is a smooth projective surface of general type. Set $L_2 := M_2|_{S_2}$. Then L_2 is a nef divisor on the surface S_2 . We have already known from the proof of Lemma 2.5 that, for $r \geq 4$,

$$h^0(X', S_2) = P_{r+2}(X) \geq \frac{1}{8}(r^2 + 6r + 8).$$

THEOREM 4.1. *Let X be a projective minimal 3-fold of general type with the canonical index $r \geq 4$. If $\dim \phi_{r+2}(X) = 3$, then $\phi_{m_1(r)}$ is stably birational where*

$$m_1(r) = \begin{cases} 16, & \text{if } r = 4 \\ 2r + 7, & \text{if } 5 \leq r \leq 6 \\ 2r + 6, & \text{if } r \geq 7. \end{cases}$$

Proof. Given an integer $t_1 > 0$, first we note that

$$|K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil + S_2 | \subset |(t_1 + 2r + 5)K_{X'}|.$$

In order to prove the birationality of ϕ_{t_1+2r+5} , it is sufficient to prove the same thing for $\Phi_{|K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil + S_2|}$.

Step 1. Reduce to the problem on a surface.

Since $K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil$ is effective by Lemma 2.5, it is enough to verify the same thing for its restriction to S_2 by virtue of (P1). The vanishing theorem gives the exact sequence

$$\begin{aligned} H^0(X', K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil + S_2) &\longrightarrow \\ H^0(S_2, K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2}) &\longrightarrow 0. \end{aligned}$$

This means

$$|K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil + S_2| \big|_{S_2} = |K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2}|.$$

And from Lemma 2.2, we have

$$|K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2} \subset |K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2}|.$$

So, sometimes, it is enough to show that $|K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2}|$ gives a birational map. Under the assumption of this theorem, it is obvious that $|L_2|$ gives a generically finite map.

Step 2. Reduce to the problem on a curve.

We suppose \overline{C} is the general member of the movable part of $|L_2|$. Since

$$K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2} = (K_{X'} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil)|_{S_2} + L_2 \geq L_2,$$

by (P1), we only have to verify the birationality of $\Phi_{|K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2}|}|_{\overline{C}}$ for a general member \overline{C} . It is obvious that

$$\begin{aligned} K_{S_2} + \lceil (t_1 + r + 2)\pi^*(K_X)^\lceil|_{S_2} &\geq K_{S_2} + \lceil t_1\pi^*(K_X)^\lceil|_{S_2} + L_2 \\ &\geq K_{S_2} + \lceil t_1\pi^*(K_X)^\lceil|_{S_2} + \overline{C}. \end{aligned}$$

It's sufficient to show that $\Phi_{|K_{S_2} + \lceil t_1\pi^*(K_X)^\lceil|_{S_2}| + \overline{C}|}|_{\overline{C}}$ is birational.

Step 3. Verifying the embedding on \overline{C} .

The vanishing theorem gives

$$|K_{S_2} + \lceil t_1\pi^*(K_X)^\lceil|_{S_2} + \overline{C}| = |K_{\overline{C}} + D_0|.$$

where $D_0 := {}^\lceil t_1 \pi^*(K_X)|_{S_2} \rceil|_{\overline{C}}$ is a divisor on \overline{C} with

$$\deg(D_0) \geq t_1 \pi^*(K_X)|_{S_2} \cdot \overline{C}.$$

It's clear that the theorem follows whenever $\deg(D_0) \geq 3$. Although, in general, $\pi^*(K_X)|_{S_2} \cdot \overline{C}$ is a positive rational number, we can estimate it in this situation.

Note that, if $|\overline{C}|$ has already given a birational map, then so does $|K_{S_2} + {}^\lceil (t_1 + r + 2) \pi^*(K_X) \rceil|_{S_2}|$ because

$$K_{S_2} + {}^\lceil (t_1 + r + 2) \pi^*(K_X) \rceil|_{S_2} \geq L_2.$$

So we may suppose that $|\overline{C}|$ gives a generically finite, non-birational map on the surface S_2 . According to Lemma 2.6(i), we get

$$\overline{C}^2 \geq 2h^0(S_2, \overline{C}) - 4 \geq \frac{1}{4}(r^2 + 6r) - 4.$$

Thus we get

$$(r+2)\pi^*(K_X)|_{S_2} \cdot \overline{C} \geq \overline{C}^2 \geq \frac{1}{4}(r^2 + 6r) - 4.$$

One can easily obtain

$$\pi^*(K_X)|_{S_2} \cdot \overline{C} \begin{cases} \geq 1, & \text{if } r = 4 \\ \geq \frac{10}{7}, & \text{if } r = 5 \\ \geq \frac{7}{4}, & \text{if } r = 6 \\ > 2, & \text{if } r \geq 7. \end{cases}$$

Thus we can see that, whenever $t_1 \geq m_1(r) - 2r - 5$,

$$\deg(D_0) \geq {}^\lceil t_1 \pi^*(K_X)|_{S_2} \cdot \overline{C} \rceil \geq 3.$$

We have proved that $\phi_{m_1(r)}$ is stably birational under assumption of the theorem. \square

THEOREM 4.2. *Let X be a projective minimal 3-fold of general type with the canonical index $r \geq 4$. If $\dim \phi_{r+2}(X) = 2$, then ϕ_{2r+6} is stably birational.*

Proof. In this case, we note that $|L_2|$ is a base point free pencil on the surface S_2 . We can write

$$L_2 \sim_{\text{lin}} \sum_{i=1}^{a_2} C_i \sim_{\text{num}} a_2 C,$$

where

$$a_2 \geq h^0(S_2, L_2) - 1 \geq \frac{1}{8}(r^2 + 6r - 8)$$

and C denotes a generic irreducible element of $|L_2|$. Given an integer $t_2 > 0$, we want to prove the birationality of ϕ_{t_2+2r+5} . For the same reason as in the proof of Theorem 4.1, it is sufficient to verify the birationality of $\Phi_{|K_{S_2} + {}^\lceil (t_2+r+2) \pi^*(K_X) \rceil|_{S_2}|}$.

Step 1. Reduce to the problem on a curve.

Since $K_{X'} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil$ is effective by Lemma 2.5, we can see that

$$K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} \geq L_2.$$

This shows that $\Phi_{|K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2}|}$ can distinguish different fibers of $\Phi_{|L_2|}$. On the other hand, we have

$$(t_2 + r + 2)\pi^*(K_X) \geq t_2\pi^*(K_X) + S_2.$$

So we get

$$K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} \geq K_{S_2} + \lceil t_2\pi^*(K_X) \rceil|_{S_2} + L_2.$$

The vanishing theorem gives the following exact sequence

$$H^0\left(S_2, K_{S_2} + \lceil t_2\pi^*(K_X) \rceil|_{S_2} + L_2\right) \longrightarrow \bigoplus_{i=1}^{a_2} H^0(C_i, K_{C_i} + D_i) \longrightarrow 0$$

where the D'_i s are divisors on the curve C_i with positive degree. This means that the map $\Phi_{|K_{S_2} + \lceil t_2\pi^*(K_X) \rceil|_{S_2} + L_2|}$ can distinguish disjoint irreducible components in a general fiber of $\Phi_{|L_2|}$. Thus the map $\Phi_{|K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2}|}$ also has this property. In order to apply (P2), we are reduced to verify that

$$\Phi_{|K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2}|} \Big|_C$$

is an embedding for a generic irreducible element C of $|L_2|$. Actually, it is sufficient to verify this property for $\Phi_{|K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2}|} \Big|_C$.

Step 2. Calculation on C .

We can write

$$(r+2)\pi^*(K_X) \sim_{\text{lin}} S_2 + \overline{E}_{\mathbb{Q}},$$

where $\overline{E}_{\mathbb{Q}}$ is an effective \mathbb{Q} -divisor. So one has

$$(r+2)\pi^*(K_X)|_{S_2} \sim_{\text{lin}} S_2|_{S_2} + E_{\mathbb{Q}} \sim_{\text{num}} a_2 C + E_{\mathbb{Q}}$$

where $E_{\mathbb{Q}} = \overline{E}_{\mathbb{Q}}|_{S_2}$ is an effective \mathbb{Q} -divisor on S_2 . Considering the system

$$\left| K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} - \frac{1}{a_2} E_{\mathbb{Q}} \right|,$$

we have

$$K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} \geq K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} - \frac{1}{a_2} E_{\mathbb{Q}}.$$

Note that

$$(t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2} E_{\mathbb{Q}} - C \sim_{\text{num}} (t_2 + r + 2) \left(1 - \frac{r+2}{a_2(t_2+r+2)} \right) \pi^*(K_X)|_{S_2},$$

which is a nef and big \mathbb{Q} -divisor on S_2 since $\beta := 1 - \frac{r+2}{a_2(t_2+r+2)} > 0$. Thus, by the vanishing theorem,

$$H^1\left(S_2, K_{S_2} + \lceil (t_2 + r + 2)\pi^*(K_X) \rceil|_{S_2} - \frac{1}{a_2} E_{\mathbb{Q}} - C\right) = 0.$$

This would give the following exact sequence

$$H^0\left(S_2, K_{S_2} + \lceil(t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\rceil\right) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := \lceil(t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\rceil|_C$. Now the main task is to show that $\deg(D) \geq 3$, which implies that $K_C + D$ is very ample since

$$\deg(K_C + D) \geq 2g(C) + 1.$$

In fact, we note that $r\pi^*(K_X)|_{S_2}$ is a nef and big Cartier divisor on S_2 , so $r\pi^*(K_X)|_{S_2} \cdot C$ is a positive integer. And we have

$$\begin{aligned} \deg(D) &\geq \left((t_2 + r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\right) \cdot C = (t_2 + r + 2)\beta\pi^*(K_X)|_{S_2} \cdot C \\ &= \left(t_2 + r + 2 - \frac{r+2}{a_2}\right)\pi^*(K_X)|_{S_2} \cdot C \\ &\geq r\pi^*(K_X)|_{S_2} \cdot C + \left(3 - \frac{r+2}{a_2}\right)\pi^*(K_X)|_{S_2} \cdot C. \end{aligned}$$

It is easy to see $3 - \frac{r+2}{a_2} > 0$. So $\deg(D) \geq 3$ follows whenever $r\pi^*(K_X)|_{S_2} \cdot C \geq 2$, which will be proved in the next step.

Step 3. Estimating $r\pi^*(K_X)|_{S_2} \cdot C$ by studying ϕ_{3r+5} .

We claim that $r\pi^*(K_X)|_{S_2} \cdot C \geq 2$. This can be derived from our studying ϕ_{3r+5} . We have to use a lot of notations to perform the calculation.

We know that

$$K_{X'} + \lceil(2r + 2)\pi^*(K_X)\rceil + S_2 \leq (3r + 5)K_{X'}.$$

The vanishing theorem gives

$$\begin{aligned} |K_{X'} + \lceil(2r + 2)\pi^*(K_X)\rceil + S_2||_{S_2} &= |K_{S_2} + \lceil(2r + 2)\pi^*(K_X)\rceil|_{S_2}| \\ \supseteq |K_{S_2} + \lceil(2r + 2)\pi^*(K_X)\rceil|_{S_2} &\supseteq |K_{S_2} + \lceil(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\rceil|. \end{aligned} \tag{4.1}$$

Because

$$(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}} - C \sim_{\text{num}} (2r + 2 - \frac{r+2}{a_2})\pi^*(K_X)|_{S_2}$$

is a nef and big \mathbb{Q} -divisor, the vanishing theorem gives

$$|K_{S_2} + \lceil(2r + 2)\pi^*(K_X)\rceil|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\rceil||_C = |K_C + D_{3r+5}|, \tag{4.2}$$

where $D_{3r+5} := \lceil(2r + 2)\pi^*(K_X)|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}}\rceil|_C$ is a divisor on C with

$$\begin{aligned} \deg(D_{3r+5}) &\geq (2r + 2 - \frac{r+2}{a_2})\pi^*(K_X)|_{S_2} \cdot C \\ &= 2r\pi^*(K_X)|_{S_2} \cdot C + (2 - \frac{r+2}{a_2})\pi^*(K_X)|_{S_2} \cdot C > 2. \end{aligned}$$

Now let M_{3r+5} be the movable part of $|(3r+5)K_{X'}|$ and M'_{3r+5} be the movable part of $|K_{X'} + \lceil (2r+2)\pi^*(K_X) \rceil + S_2|$. Then it's clear that

$$(3r+5)\pi^*(K_X) \geq M_{3r+5} \geq M'_{3r+5}.$$

Let L_{3r+5} be the movable part of $|K_{S_2} + \lceil (2r+2)\pi^*(K_X) \rceil|_{S_2}|$ and L'_{3r+5} be the movable part of

$$|K_{S_2} + \lceil (2r+2)\pi^*(K_X) \rceil|_{S_2} - \frac{1}{a_2}E_{\mathbb{Q}} \rceil.$$

Then $L_{3r+5} \geq L'_{3r+5}$. From (4.1) and Lemma 2.7, we have $M_{3r+5}|_{S_2} \geq L_{3r+5}$. From (4.2) and Lemma 2.7, we have

$$\begin{aligned} h^0(C, L'_{3r+5}|_C) &= h^0(C, K_C + D_{3r+5}) \\ &= g(C) - 1 + \deg(D_{3r+5}) \geq g(C) + 2. \end{aligned}$$

Using R-R and the Clifford's theorem and noting that $g(C) \geq 2$, one can easily see that $L'_{3r+5} \cdot C \geq 2g(C) + 1 \geq 5$. So we get

$$\begin{aligned} (3r+5)\pi^*(K_X)|_{S_2} \cdot C &\geq M_{3r+5}|_{S_2} \cdot C \\ &\geq L_{3r+5} \cdot C \geq L'_{3r+5} \cdot C \geq 5 \end{aligned}$$

i.e. $r\pi^*(K_X)|_{S_2} \cdot C \geq \frac{5r}{3r+5} > 1$. Noting that $r\pi^*(K_X)|_{S_2} \cdot C$ is an integer, we see $r\pi^*(K_X)|_{S_2} \cdot C \geq 2$. The proof is completed. \square

From Theorem 4.1 and Theorem 4.2, we instantly have the following

COROLLARY 4.3. *Let X be a projective minimal 3-fold of general type with the canonical index $r \geq 6$. Then $\phi_{m_0(r)}$ is stably birational.*

Proof. The main point is $\dim \phi_{r+2}(X) \geq 2$ for $r \geq 6$ according to Lemma 2.4. \square

For $4 \leq r \leq 5$, we have to treat the case with $\dim \phi_{r+2}(X) = 1$. We shall use a similar method as above by studying the system $|(r+3)K_{X'}|$ because $\dim \phi_{r+3}(X) \geq 2$. First we take a birational modification $\pi : X' \rightarrow X$, according to Hironaka, such that

- (i) X' is smooth;
- (ii) the system $|(r+3)K_{X'}|$ defines a morphism;
- (iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by M_3 the movable part of $|(r+3)K_{X'}|$. Take a general member $S_3 \in |M_3|$. Then S_3 is a smooth irreducible projective surface of general type. Set $L_3 := M_3|_{S_3}$. Then L_3 is a nef divisor on the surface S_3 . Taking $s = 3$ and using (2.1) and (2.2), we have

$$h^0(X', S_3) = P_{r+3}(X) \geq \frac{1}{8}(r^2 + 10r + 24).$$

Thus $h^0(S_3, L_3) \geq h^0(X', S_3) - 1 \geq \frac{1}{8}(r^2 + 10r + 16)$.

THEOREM 4.4. *Let X be a projective minimal 3-fold of general type with the canonical index $r = 4, 5$. Suppose $\dim \phi_{r+3}(X) = 3$. Then*

- (i) *if $r = 4$, ϕ_{17} is stably birational;*
- (ii) *if $r = 5$, ϕ_{18} is stably birational.*

Proof. Given an integer $t_3 > 0$, we note that

$$|K_{X'} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil + S_3| \subset |(t_3 + 2r + 7)K_{X'}|.$$

In order to prove the birationality of ϕ_{t_3+2r+7} , it is sufficient to prove the same thing for $\Phi_{|K_{X'} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil + S_3|}$.

Step 1. Reduce to the problem on a surface.

Since $K_{X'} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil$ is effective by Lemma 2.5, it is enough to verify the same thing for its restriction to S_3 by virtue of (P1). The vanishing theorem gives

$$|K_{X'} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil + S_3| \big|_{S_3} = |K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3}|.$$

And from Lemma 2.2, we have

$$|K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3} \subset |K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3}|.$$

So, sometimes, it is enough to show that $|K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3}|$ gives a birational map. Under the assumption of this theorem, it is obvious that $|L_3|$ gives a generically finite map.

Step 2. Reduce to the problem on a curve.

We suppose C' is the general member of the movable part of $|L_3|$. Since

$$K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3} = (K_{X'} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil)|_{S_3} + L_3 \geq L_3,$$

by (P1), we only have to verify the birationality of $\Phi_{|K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3}|} \big|_{C'}$ for a general member C' . It is obvious that

$$\begin{aligned} K_{S_3} + \lceil (t_3 + r + 3)\pi^*(K_X) \rceil|_{S_3} &\geq K_{S_3} + \lceil t_3\pi^*(K_X) \rceil|_{S_3} + L_3 \\ &\geq K_{S_3} + \lceil t_3\pi^*(K_X) \rceil|_{S_3} + C'. \end{aligned}$$

It's sufficient to show that $\Phi_{|K_{S_3} + \lceil t_3\pi^*(K_X) \rceil|_{S_3} + C'|} \big|_{C'}$ is birational.

Step 3. Verifying the embedding on C' .

The vanishing theorem gives

$$|K_{S_3} + \lceil t_3\pi^*(K_X) \rceil|_{S_3} + C' \big|_{C'} = |K_{C'} + D_1|,$$

where $D_1 := \lceil t_3\pi^*(K_X) \rceil|_{S_3}|$ is a divisor on C' with

$$\deg(D_1) \geq t_3\pi^*(K_X)|_{S_3} \cdot C'.$$

It's clear that the theorem follows whenever $\deg(D_1) \geq 3$. Although, in general, $\pi^*(K_X)|_{S_3} \cdot C'$ is only a rational number, we can still estimate it in this situation.

Note that, if $|C'|$ has already given a birational map, then so does $|K_{S_3} + {}^\Gamma(t_3 + r + 3)\pi^*(K_X)^\Gamma|_{S_3}|$ because

$$K_{S_3} + {}^\Gamma(t_3 + r + 3)\pi^*(K_X)^\Gamma|_{S_3} \geq L_3.$$

So we may suppose that $|C'|$ gives a generically finite, non-birational map on the surface S_3 . According to Lemma 2.6(i), we get

$$C'^2 \geq 2h^0(S_2, C') - 4 \geq \frac{1}{4}(r^2 + 10r).$$

Thus we get

$$(r + 3)\pi^*(K_X)|_{S_3} \cdot C' \geq C'^2 \geq \frac{1}{4}(r^2 + 10r).$$

One can easily obtain

$$\pi^*(K_X)|_{S_3} \cdot C' \begin{cases} \geq 2, & \text{if } r = 4 \\ \geq \frac{19}{8}, & \text{if } r = 5. \end{cases}$$

Thus we can see that, whenever $t_3 \geq 2$ if $r = 4$ or $t_3 \geq 1$ if $r = 5$,

$$\deg(D_1) \geq {}^\Gamma t_3 \pi^*(K_X)|_{S_3} \cdot C'^\Gamma \geq 3.$$

We have proved the theorem. \square

THEOREM 4.5. *Let X be a projective minimal 3-fold of general type with the canonical index $r = 4, 5$. Suppose $\dim \phi_{r+3}(X) = 2$. Then*

- (i) if $r = 4$, ϕ_{16} is stably birational;
- (ii) if $r = 5$, ϕ_{18} is stably birational.

Proof. In this case, we note that $|L_3|$ is a base point free pencil on the surface S_3 .

We can write

$$L_3 \sim_{\text{lin}} \sum_{i=1}^{a_3} C_i \sim_{\text{num}} a_3 C,$$

where

$$a_3 \geq h^0(S_3, L_3) - 1 \geq \frac{1}{8}(r^2 + 10r + 8)$$

and C denotes a generic irreducible element of $|L_3|$. Given an integer $t_4 > 0$, we want to prove the birationality of ϕ_{t_4+2r+7} . For the same reason as in the proof of Theorem 4.4, it is sufficient to verify the birationality of $\Phi_{|K_{S_3} + {}^\Gamma(t_4+r+3)\pi^*(K_X)^\Gamma|_{S_3}|}$.

Step 1. Reduce to the problem on a curve.

Since $K_{X'} + {}^\Gamma(t_4 + r + 3)\pi^*(K_X)^\Gamma$ is effective by Lemma 2.5, we can see that

$$K_{S_3} + {}^\Gamma(t_4 + r + 3)\pi^*(K_X)^\Gamma|_{S_3} \geq L_3.$$

This shows that $\Phi_{|K_{S_3} + {}^\Gamma(t_4+r+3)\pi^*(K_X)^\Gamma|_{S_3}|}$ can distinguish different fibers of $\Phi_{|L_3|}$. On the other hand, we have

So we get

$$K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} \geq K_{S_3} + \lceil t_4\pi^*(K_X) \rceil|_{S_3} + L_3.$$

The vanishing theorem gives the following exact sequence

$$H^0\left(S_3, K_{S_3} + \lceil t_4\pi^*(K_X) \rceil|_{S_3} + L_3\right) \longrightarrow \bigoplus_{i=1}^{a_3} H^0(C_i, K_{C_i} + D_i) \longrightarrow 0$$

where the D'_i s are divisors on the curve C_i with positive degree. This means that the map $\Phi_{|K_{S_3} + \lceil t_4\pi^*(K_X) \rceil|_{S_3} + L_3|}$ can distinguish disjoint irreducible components in a general fiber of $\Phi_{|L_3|}$. Thus the map $\Phi_{|K_{S_3} + \lceil (t_4+r+3)\pi^*(K_X) \rceil|_{S_3}|}$ also has this property. In order to apply (P2), we are reduced to verify that

$$\Phi_{|K_{S_3} + \lceil (t_4+r+3)\pi^*(K_X) \rceil|_{S_3}|} \Big|_C$$

is an embedding for a generic irreducible element C of $|L_3|$. Actually, it is sufficient to verify this property for $\Phi_{|K_{S_3} + \lceil (t_4+r+3)\pi^*(K_X) \rceil|_{S_3}|} \Big|_C$.

Step 2. Calculation on C .

We can write

$$(r+3)\pi^*(K_X) \sim_{\text{lin}} S_3 + \overline{E_{\mathbb{Q}}},$$

where $\overline{E_{\mathbb{Q}}}$ is an effective \mathbb{Q} -divisor. So one has

$$(r+3)\pi^*(K_X)|_{S_3} \sim_{\text{lin}} S_3|_{S_3} + E_{\mathbb{Q}} \sim_{\text{num}} a_3 C + E_{\mathbb{Q}}$$

where $E_{\mathbb{Q}} = \overline{E_{\mathbb{Q}}}|_{S_3}$ is an effective \mathbb{Q} -divisor on S_3 . Considering the system

$$\left| K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} - \frac{1}{a_3} E_{\mathbb{Q}} \right|,$$

we have

$$K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} \geq K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} - \frac{1}{a_3} E_{\mathbb{Q}}.$$

Note that

$$(t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3} E_{\mathbb{Q}} - C \sim_{\text{num}} (t_4 + r + 3) \left(1 - \frac{r+3}{a_3(t_4+r+3)} \right) \pi^*(K_X)|_{S_3},$$

which is a nef and big \mathbb{Q} -divisor on S_3 since $\gamma := 1 - \frac{r+3}{a_3(t_4+r+3)} > 0$. Thus, by the vanishing theorem,

$$H^1\left(S_3, K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} - \frac{1}{a_3} E_{\mathbb{Q}} - C\right) = 0.$$

This would give the following exact sequence

$$H^0\left(S_3, K_{S_3} + \lceil (t_4 + r + 3)\pi^*(K_X) \rceil|_{S_3} - \frac{1}{a_3} E_{\mathbb{Q}}\right) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := \lceil (t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \rceil|_C$. Now the main task is to show that $\deg(D) \geq 3$, which implies that $K_C + D$ is very ample since

$$\deg(K_C + D) \geq 2g(C) + 1.$$

In fact, we note that $r\pi^*(K_X)|_{S_3}$ is a nef and big Cartier divisor on S_3 , so $r\pi^*(K_X)|_{S_3} \cdot C$ is a positive integer. And we have

$$\begin{aligned} \deg(D) &\geq \left((t_4 + r + 3)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \right) \cdot C = (t_4 + r + 3)\gamma\pi^*(K_X)|_{S_3} \cdot C \\ &= \left(t_4 + r + 3 - \frac{r+3}{a_3} \right) \pi^*(K_X)|_{S_3} \cdot C \\ &\geq r\pi^*(K_X)|_{S_3} \cdot C + \left(4 - \frac{r+3}{a_3} \right) \pi^*(K_X)|_{S_3} \cdot C. \end{aligned}$$

It is easy to see $4 - \frac{r+3}{a_3} > 0$. So $\deg(D) \geq 3$ follows whenever $r\pi^*(K_X)|_{S_3} \cdot C \geq 2$, which will be proved in the next step.

Step 3. Estimating $r\pi^*(K_X)|_{S_3} \cdot C$ by studying ϕ_{3r+5} .

We claim that $r\pi^*(K_X)|_{S_3} \cdot C \geq 2$. This can be derived from our studying ϕ_{3r+5} . We have to use a lot of notations to perform the calculation.

We know that

$$K_{X'} + \lceil (2r+1)\pi^*(K_X) \rceil + S_3 \leq (3r+5)K_{X'}.$$

The vanishing theorem gives

$$\begin{aligned} |K_{X'} + \lceil (2r+1)\pi^*(K_X) \rceil + S_3|_{S_3} &= |K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil|_{S_3}| \\ &\supseteq |K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil|_{S_3} \supseteq |K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \rceil|. \end{aligned} \tag{4.3}$$

Because

$$(2r+1)\pi^*(K_X)|_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} - C \sim_{\text{num}} (2r+1 - \frac{r+3}{a_3})\pi^*(K_X)|_{S_3}$$

is a nef and big \mathbb{Q} -divisor, the vanishing theorem gives

$$|K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil|_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \rceil|_C = |K_C + D_{3r+5}|, \tag{4.4}$$

where $D_{3r+5} := \lceil (2r+1)\pi^*(K_X) \rceil_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \rceil|_C$ is a divisor on C with

$$\begin{aligned} \deg(D_{3r+5}) &\geq (2r+1 - \frac{r+3}{a_3})\pi^*(K_X)|_{S_3} \cdot C \\ &= 2r\pi^*(K_X)|_{S_3} \cdot C + \left(1 - \frac{r+3}{a_3} \right) \pi^*(K_X)|_{S_3} \cdot C > 2. \end{aligned}$$

Now let M_{3r+5} be the movable part of $|(3r+5)K_{X'}|$ and M'_{3r+5} be the movable part of $|K_{X'} + \lceil (2r+1)\pi^*(K_X) \rceil + S_3|$. Then it's clear that

$$(2r+5)\pi^*(K_{X'}) \supseteq M_{3r+5} \supseteq M'_{3r+5}$$

Let L_{3r+5} be the movable part of $|K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil|_{S_3}|$ and L'_{3r+5} be the movable part of

$$|K_{S_3} + \lceil (2r+1)\pi^*(K_X) \rceil_{S_3} - \frac{1}{a_3}E_{\mathbb{Q}} \rceil|.$$

Then $L_{3r+5} \geq L'_{3r+5}$. From (4.3) and Lemma 2.7, we have $M_{3r+5}|_{S_3} \geq L_{3r+5}$. From (4.4) and Lemma 2.7, we have

$$\begin{aligned} h^0(C, L'_{3r+5}|_C) &= h^0(C, K_C + D_{3r+5}) \\ &= g(C) - 1 + \deg(D_{3r+5}) \geq g(C) + 2. \end{aligned}$$

Using R-R and the Clifford's theorem and noting that $g(C) \geq 2$, one can easily see that $L'_{3r+5} \cdot C \geq 2g(C) + 1 \geq 5$. So we get

$$\begin{aligned} (3r+5)\pi^*(K_X)|_{S_3} \cdot C &\geq M_{3r+5}|_{S_3} \cdot C \\ &\geq L_{3r+5} \cdot C \geq L'_{3r+5} \cdot C \geq 5 \end{aligned}$$

i.e. $r\pi^*(K_X)|_{S_3} \cdot C \geq \frac{5r}{3r+5} > 1$. Noting that $r\pi^*(K_X)|_{S_3} \cdot C$ is an integer, we see $r\pi^*(K_X)|_{S_3} \cdot C \geq 2$. The proof is completed. \square

From Theorem 3.1, we can take $m_0(2) = 11$ and $m_0(3) = 15$. Theorems 4.4 and 4.5 imply $m_0(4) = 17$ and $m_0(5) = 18$. Therefore the main theorem follows.

5. Threefolds with positive geometric genus

Throughout this section, we still suppose X is a projective minimal 3-fold of general type. Our aim is to study 3-folds with big geometric genus using the method of the Main Theorem. Kollar (Corollary 4.8 of [11]) proved that ϕ_{16} is birational if $p_g(X) \geq 2$. Reviewing the parallel results on surfaces and Gorenstein 3-folds, one should expect a better bound for the birationality of ϕ_m .

To begin the argument, we first take a birational modification $\pi : X' \rightarrow X$ according to Hironaka such that

- (i) X' is smooth;
- (ii) $|K_{X'}|$ gives a morphism;
- (iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Set $g := \phi_1 \circ \pi$ and take the Stein-factorization of

$$g : X' \xrightarrow{f} W \rightarrow W' \subset \mathbb{P}^N$$

where W' is the image of X' through g and f is a fibration. Let M be the movable part of $|K_{X'}|$. We can write

$$K_{X'} \sim_{\text{lin}} M + E' \quad \text{and} \quad \pi^*(K_X) =_{\mathbb{Q}} M + \overline{E}_{\mathbb{Q}}$$

where E' is an effective divisor and $\overline{E}_{\mathbb{Q}}$ is an effective \mathbb{Q} -divisor.

PROPOSITION 5.1. *Let X be a minimal projective threefold of general type. If $\dim\phi_1(X) \geq 2$, then*

- (i) ϕ_4 is generically finite;
- (ii) ϕ_3 is generically finite provided $p_g(X) \geq 4$.

Proof. Because $p_g(X) > 0$, it is sufficient to prove for the case when $\dim\phi_1(X) = 2$. Let $S \in |M|$ be the general member. Then S is a smooth projective surface of general type. We have $S|_S \sim_{\text{num}} aC$, where C is a smooth curve and $a \geq p_g(X) - 2$. Considering the system $|K_{X'} + {}^{\Gamma}\pi^*(K_X)^\Gamma + 2S|$, we have

$$|K_{X'} + {}^{\Gamma}\pi^*(K_X)^\Gamma + 2S| |_S = |K_S + {}^{\Gamma}\pi^*(K_X)^\Gamma|_S + S|_S|.$$

Besides, we have

$$|K_S + {}^{\Gamma}\pi^*(K_X)|_S^\Gamma + S|_S| \Big|_C = |K_C + D|,$$

where $D := {}^{\Gamma}\pi^*(K_X)|_S^\Gamma$ is a divisor on C of positive degree. Thus $|K_C + D|$ gives a finite map and so does ϕ_4 . This derives (i). If $p_g(X) \geq 4$, then $a \geq 2$. By the vanishing theorem, we get

$$\begin{aligned} |K_{X'} + {}^{\Gamma}\pi^*(K_X)^\Gamma + S| |_S &= |K_S + {}^{\Gamma}\pi^*(K_X)^\Gamma|_S| \\ &\supset |K_S + {}^{\Gamma}\pi^*(K_X)|_S^\Gamma|. \end{aligned}$$

We can write $\pi^*(K_X)|_S = S|_S + E_{\mathbb{Q}}$, where $E_{\mathbb{Q}}$ is an effective \mathbb{Q} -divisor. It is obvious that

$$\pi^*(K_X)|_S - \frac{1}{a}E_{\mathbb{Q}} - C \sim_{\text{num}} \left(1 - \frac{1}{a}\right)\pi^*(K_X)|_S.$$

So

$$H^1\left(S, K_S + {}^{\Gamma}\pi^*(K_X)|_S - \frac{1}{a}E_{\mathbb{Q}}^\Gamma - C\right) = 0,$$

which gives

$$|K_S + {}^{\Gamma}\pi^*(K_X)|_S - \frac{1}{a}E_{\mathbb{Q}}^\Gamma| \Big|_C = |K_C + D|,$$

where $D := {}^{\Gamma}\pi^*(K_X) - \frac{1}{a}E_{\mathbb{Q}}^\Gamma|_C$ is a divisor on C of positive degree. Thus $|K_C + D|$ gives a generically finite map. By (P2), ϕ_3 is also generically finite. \square

PROPOSITION 5.2. *Let X be a minimal projective threefold of general type. If $\dim\phi_1(X) \geq 2$, then*

- (i) ϕ_8 is birational;
- (ii) ϕ_6 is birational provided $p_g(X) \geq 4$.

Proof. If $\dim\phi_1(X) = 3$, then it is very easy to prove the birationality of ϕ_6 by standard argument. We mainly discuss the case when $\dim\phi_1(X) = 2$. To prove (i), let M_4 be the movable part of $|4K_{X'}|$. We can modify π , if necessary, such that $|M_4|$ is also base point free. We have

$$|K_{X'} + {}^{\Gamma}\pi^*(K_X)^\Gamma + M_4 + 2S| |_S = |K_S + {}^{\Gamma}\pi^*(K_X)^\Gamma|_S + L_4 + S|_S|,$$

where $L_4 := M_4|_S$ is nef and $\Phi_{|L_4|}$ is generically finite. It is not difficult to see that the right system gives a birational map using the method which has been applied frequently in this paper. So ϕ_8 is birational according to (P1).

If $p_g(X) \geq 4$, denote by M_3 the moving part of $|3K_{X'}|$. For the same reason, we can suppose $|M_3|$ is also base point free. We have

$$\begin{aligned} |K_{X'} + {}^{\lceil}4\pi^*(K_X)\rceil + S| |_S &= |K_S + {}^{\lceil}4\pi^*(K_X)\rceil|_S| \\ &\supset |K_S + {}^{\lceil}4\pi^*(K_X)|_S\rceil \supset |K_S + {}^{\lceil}\pi^*(K_X)\rceil|_S\rceil + L_3|. \end{aligned}$$

where $L_3 := M_3|_S$, which is a nef and big divisor on S . And $|L_3|$ gives a generically finite map. Because

$$\pi^*(K_X)|_S - C - \frac{1}{a}E_{\mathbb{Q}}$$

is nef and big, the vanishing theorem will imply that

$$|K_S + {}^{\lceil}\pi^*(K_X)\rceil|_S - \frac{1}{a}E_{\mathbb{Q}}\rceil + L_3| |_C = |K_C + L_3|_C + D|,$$

where $D = {}^{\lceil}\pi^*(K_X)\rceil|_S - \frac{1}{a}E_{\mathbb{Q}}\rceil|_C$ is a divisor of positive degree. Thus $\deg(L_3|_C + D) \geq 3$ and $K_C + L_3|_C + D$ is very ample. This shows that ϕ_6 is birational. \square

PROPOSITION 5.3. *Let X be a minimal projective threefold of general type. If $\dim\phi_1(X) = 1$, then*

- (i) ϕ_9 is birational;
- (ii) ϕ_6 is birational provided $p_g(X) \geq 12$.

Proof. (i). In this case, W is a nonsingular curve. We set $b := g(W)$, the genus of W .

If $b > 0$, then ϕ_1 is actually a morphism. In this case, there is no need to make the modification π , i.e. $X' = X$. Though K_X is not Cartier, it is a Weil divisor. We can still define the system $|K_X|$ in a natural way. We have $M \sim_{\text{lin}} \sum_{i=1}^a S_i$, where the S_i are fibers of f . Noting that the singularities on X are all isolated, a general S_i is a smooth projective surface of general type. Using Kawamata's vanishing theorem ([10]) for \mathbb{Q} -Cartier Weil divisors, we have $H^1(X, \omega_X^{[k]}) = 0$ whenever $k > 1$. Thus $\phi_6|_{S_i} = \Phi_{|5K_{S_i}|}$ is birational. According to (P2), ϕ_6 is birational.

If $b = 0$, $W = \mathbb{P}^1$ and we have the fibration $f : X' \rightarrow \mathbb{P}^1$. Let S be a general fiber of f . Then S is a smooth projective surface of general type. We divide S into two categories:

- (I) $(K_{S_0}^2, p_g(S)) \neq (1, 2)$ and $(2, 3)$;
- (II) the rest.

where S_0 denotes the minimal model of S .

Suppose S is of type (I). There is a common property for these surfaces that the 3-canonical maps are birational. To deal with this situation, We can use Kollar's technique (the proof of Corollary 4.8 in [11]). Because $p_g(X) > 0$, we have $p_g(S) > 0$. Let $\sigma : S \rightarrow S_0$ be the contraction onto the minimal model. According to Theorem 3.1 in [5], we see that $|2K_{S_0}|$ is base point free. So the movable part of $|2K_S|$ is $\sigma^*(2K_{S_0})$. We have $H^0(\omega_{X'}^7) = H^0(\mathbb{P}^1, f_*\omega_{X'}^7)$ and an injection $\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$, and hence an injection $\mathcal{O}(5) \hookrightarrow f_*\omega_{X'}^5$. This gives an injection

$$\mathcal{O}(5) \otimes f_*\omega_{X'}^2 \hookrightarrow f_*\omega_{X'}^7,$$

where

$$\mathcal{O}(5) \otimes f_*\omega_{X'}^2 = \mathcal{O}(1) \otimes f_*\omega_{X'}^2$$

It is well-known that $f_*\omega_{X'/\mathbb{P}^1}^2$ is a sum of line bundles of non-negative degree on \mathbb{P}^1 . The local sections of $f_*\omega_{X'}^2$ give the bicanonical map for S , and all these extend to global sections of $\mathcal{O}(5) \otimes f_*\omega_{X'}^2$. Moreover the sections of $\mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^2$ separate different fibers. Suppose M_7 is the movable part of $|7K_{X'}|$. Because $\phi_7 = \Phi_{|M_7|}$, we can see from the above argument that $M_7|_S \geq \sigma^*(2K_{S_0})$. Now considering the system $|K_{X'} + \lceil 7\pi^*(K_X)^\lceil + S |$, we have

$$\begin{aligned} |K_{X'} + \lceil 7\pi^*(K_X)^\lceil + S | |_S &= |K_S + \lceil 7\pi^*(K_X)^\lceil|_S | \\ &\supset |K_S + M_7|_S \supset |K_S + \sigma^*(2K_{S_0})|. \end{aligned}$$

Because $|K_S + \sigma^*(2K_{S_0})|$ gives a birational map, we see that ϕ_9 is birational.

Suppose S is of type (II) and $(K_{S_0}^2, p_g(S)) = (2, 3)$. We want to show that ϕ_8 is birational. We know that the movable part of $|K_{X'}|$ is linearly equivalent to a disjoint union of irreducible copies of S . We have

$$|K_{X'} + \lceil \pi^*(K_X)^\lceil + S | |_S = |K_S + \lceil \pi^*(K_X)^\lceil|_S \supset |K_S|.$$

Because the movable part of $|K_S|$ gives a finite map onto \mathbb{P}^2 , we can see that ϕ_3 is generically finite. Denote by M_3 the movable part of $|3K_{X'}|$ and by M_5 the movable part of $|5K_{X'}|$. In order to prove the birationality of ϕ_8 , we should study $|M_5|_S |$. We have $K_{X'} + \lceil 3\pi^*(K_X)^\lceil + S \leq 5K_{X'}$. The vanishing theorem gives

$$|K_{X'} + \lceil 3\pi^*(K_X)^\lceil + S | |_S = |K_S + \lceil 3\pi^*(K_X)^\lceil|_S |.$$

We suppose that K_0 is the movable part of $|K_S|$. Denote by M_+ the movable part of $|K_{X'} + \lceil 3\pi^*(K_X)^\lceil + S |$. Then $M_+ \leq M_5$. By Lemma 2.7, we can see that $M_+|_S$ contains the movable part of $|K_S + \lceil 3\pi^*(K_X)^\lceil|_S |$. Let L be the movable part of $|M_3|_S |$. Then $K_S + \lceil 3\pi^*(K_X)^\lceil|_S \geq K_0 + L$ and $K_0 + L$ is movable. So $M_+|_S \geq K_0 + L$. On the other hand, because $M_5 \geq M_+$, so $M_5|_S \geq K_0 + L$. Now it is time to study the ϕ_8 . For an obvious reason, we can suppose that $|M_5|$ is base point free. This assumption means that M_5 is nef and big. The vanishing theorem gives the exact sequence

$$H^0(X', K_{X'} + \lceil \pi^*(K_X)^\lceil + M_5 + S) \longrightarrow H^0(S, K_S + \lceil \pi^*(K_X)^\lceil|_S + M_5|_S) \longrightarrow 0.$$

We note that

$$K_S + \lceil \pi^*(K_X)^\lceil|_S + M_5|_S \geq K_S + \lceil \pi^*(K_X)|_S^\lceil + L + K_0.$$

Since $\pi^*(K_X)|_S$ is nef and big, $\lceil \pi^*(K_X)|_S^\lceil$ is effective, $\dim \Phi_{|K_0|}(S) = 2$ and $\dim \Phi_{|L|}(S) = 2$, using our method again, it is easy to see that $|K_S + \lceil \pi^*(K_X)|_S^\lceil + L + K_0|$ gives a birational map. Which shows that

$$|K_{X'} + \lceil \pi^*(K_X)^\lceil + M_5 + S |$$

gives a birational map and so does ϕ_8 .

Suppose S is of type (II) and $(K_{S_0}^2, p_g(S)) = (1, 2)$. We want to show that ϕ_9 is birational. This is the most frustrating case because $\Phi_{|4K_S|}$ is not birational. We recall that $|K_S|$ has no fixed component, that it has exactly one base point

and that a general member of this system is a smooth irreducible curve of genus 2. Thus the movable part C of $|K_S|$ is also a smooth curve of genus two. Furthermore $C \leq \sigma^*(K_{S_0})$. Because

$$|K_{X'} + \lceil \pi^*(K_X)^\lceil + S \rceil|_S = |K_S + \lceil \pi^*(K_X)^\lceil|_S| \supset |K_S| \supset |C|,$$

we see that $\dim \phi_3(X) \geq 2$. We still denote by M_3 the movable part of $|3K_{X'}|$ and by M'_3 the movable part of $|K_{X'} + \lceil \pi^*(K_X)^\lceil + S \rceil|$. According to Lemma 2.7, $M_3|_S \geq M'_3|_S \geq C$. Now we consider the system

$$|K_{X'} + \lceil 4\pi^*(K_X)^\lceil + M_3 + S \rceil|.$$

Actually we can take further modification to π such that $|M_3|$ is also base point free. This means we can suppose M_3 is nef. By the Kawamata-Viehweg vanishing theorem, we have

$$\begin{aligned} |K_{X'} + \lceil 4\pi^*(K_X)^\lceil + M_3 + S \rceil|_S &= |K_S + \lceil 4\pi^*(K_X)^\lceil|_S + M_3|_S| \\ &\supset |K_S + \lceil 4\pi^*(K_X)^\lceil|_S + C|. \end{aligned}$$

We can use the vanishing theorem once more so that we get

$$|K_S + \lceil 4\pi^*(K_X)^\lceil|_S + C|_C = |K_C + D|,$$

where $D := \lceil 4\pi^*(K_X)^\lceil|_S \rceil_C$. So if we can prove $4\pi^*(K_X)|_S \cdot C > 2$, then $\deg(D) \geq 3$ which induces the birationality of ϕ_9 , because

$$|K_{X'} + \lceil 4\pi^*(K_X)^\lceil + M_3 + S \rceil| \subset |9K_{X'}|.$$

Thus we only have to prove the following claim.

Claim. $\xi := \pi^*(K_X)|_S \cdot C \geq \frac{3}{5}$.

The idea is to find an initial estimation to ξ by first proving that ϕ_{10} is birational. Then we can optimize this estimation by an infinite programme. We will find that the limit estimation is $\frac{3}{5}$. Actually our second estimation is enough for us to show the birationality of ϕ_9 . We present a better estimation here hoping that it might be useful to prove the birationality of ϕ_8 in future.

From now on, we prove the claim. Let M_5 , M_7 and M_{10} be the movable part of $|5K_{X'}|$, $|7K_{X'}|$ and $|10K_{X'}|$ respectively. For the same reason, we can suppose they are all nef. We can see that

$$\begin{aligned} |K_{X'} + \lceil 3\pi^*(K_X)^\lceil + S \rceil|_S &= |K_S + \lceil 3\pi^*(K_X)^\lceil|_S| \\ &\supset |K_S + M_3|_S \supset |K_S + C|. \end{aligned}$$

Because $q(S) = 0$, $|K_S + C|$ gives a generically finite map. So ϕ_5 is generically finite. Suppose L_5 is the movable part of $|M_5|_S$. Then $\dim \Phi_{|L_5|}(S) = 2$. Therefore we can see that $L_5 \cdot C \geq 2$ for a general element C . We also have

$$|K_{X'} + \lceil 5\pi^*(K_X)^\lceil + S \rceil|_S \supset |K_S + \lceil 5\pi^*(K_X)^\lceil|_S| \supset |C + L_5|.$$
(5.1)

Noting that $C + L_5$ is movable and by Lemma 2.7, we see that $M'_7|_S \geq C + L_5$ where M'_7 is the movable part of $|K_{X'} + {}^{\lceil} \pi^*(K_X) \rceil + S|$. Because $M_7 \geq M'_7$, $M_7|_S \geq C + L_5$. We have

$$\begin{aligned} |K_{X'} + {}^{\lceil} \pi^*(K_X) \rceil + M_7 + S| |_S &= |K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + M_7|_S| \\ &\supset |K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_5 + C|. \end{aligned} \quad (5.2)$$

Now it is obvious that

$$|K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_5 + C| |_C = |K_C + G|,$$

where $G := ({}^{\lceil} \pi^*(K_X) \rceil|_S + L_5)|_C$ is a divisor of degree ≥ 3 and so $h^0(C, K_C + G) \geq g(C) + 2 = 4$. Suppose L_{10} is the movable part of $|M_{10}|_S|$ and M'_{10} is the movable part of $|K_{X'} + {}^{\lceil} \pi^*(K_X) \rceil + M_7 + S|$. Let L'_{10} be the movable part of

$$|K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_5 + C|.$$

By Lemma 2.7, we have $M'_{10}|_S \geq L'_{10}$ and

$$h^0(C, L'_{10}|_C) = h^0(K_C + G) \geq 4.$$

Noting that $M_{10} \geq M'_{10}$, we have $L_{10} \geq L'_{10}$ and $h^0(C, L_{10}|_C) \geq 4$. Because C is a curve of genus 2, by R-R, we see that $L_{10} \cdot C \geq 5$. This means that

$$10\pi^*(K_X)|_S \cdot C \geq M_{10}|_S \cdot C \geq L_{10} \cdot C \geq 5.$$

So we get $\xi \geq \frac{1}{2}$.

Suppose M_{12} is the movable part of $|12K_{X'}|$. Similar to (5.1), we have

$$\begin{aligned} |K_{X'} + {}^{\lceil} 10\pi^*(K_X) \rceil + S| |_S &= |K_S + {}^{\lceil} 10\pi^*(K_X) \rceil|_S| \\ &\supset |K_S + L_{10}| \supset |C + L_{10}|. \end{aligned}$$

Using Lemma 2.7, we can easily see that $M_{12}|_S \geq C + L_{10}$. Replacing M_7 by M_{12} in (5.2), we also have

$$\begin{aligned} |K_{X'} + {}^{\lceil} \pi^*(K_X) \rceil + M_{12} + S| |_S &= |K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + M_{12}|_S| \\ &\supset |K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_{10} + C|. \end{aligned}$$

Using the vanishing theorem once more, we have

$$|K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_{10} + C| |_C = |K_C + {}^{\lceil} \pi^*(K_X) \rceil|_C + L_{10}|_C|.$$

Let L'_{15} be the movable part of $|K_S + {}^{\lceil} \pi^*(K_X) \rceil|_S + L_{10} + C|$. From Lemma 2.7, it is easy to see $M_{15}|_S \geq L'_{15}$ and

$$h^0(C, L'_{15}|_C) = h^0(K_C + {}^{\lceil} \pi^*(K_X) \rceil|_C + L_{10}|_C)$$

$$\geq \deg({}^{\lceil} \pi^*(K_X) \rceil|_C) + \deg(L_{10}|_C) + g(C) - 1 \geq 7$$

where M_{15} is the movable part of $|15K_{X'}|$. By R-R and the Clifford's theorem, we see that

$$L'_{15} \cdot C \geq h^0(C, L'_{15}|_C) + g(C) - 1 \geq 8.$$

Thus

$$15\pi^*(K_X)|_S \cdot C \geq M_{15}|_S \cdot C \geq L'_{15} \cdot C \geq 8.$$

This means that $\xi \geq \frac{8}{15} > \frac{1}{2}$, which directly induces the birationality of ϕ_9 .

We can infinitely repeat this programme, but omit the details. So we can get the following sequence

$$\begin{aligned} n_0 &= 10, \quad d_0 = 5 \\ n_1 &= n_0 + 5, \quad d_1 = d_0 + 3 \\ &\dots \\ n_k &= n_{k-1} + 5, \quad d_k = d_{k-1} + 3 \\ &\dots \\ \xi &\geq \frac{d_k}{n_k}, \text{ for all } k. \end{aligned}$$

Thus

$$\xi \geq \lim_{k \rightarrow \infty} \frac{d_k}{n_k} = \lim_{k \rightarrow \infty} \frac{3k+5}{5k+10} = \frac{3}{5}.$$

The claim is proved.

(ii). If $p_g(X) \geq 12$, then we have $\mathcal{O}(11) \hookrightarrow f_*\omega_{X'}$. So

$$\mathcal{O}(1) \otimes f_*\omega_{X'/\mathbb{P}^1}^5 = \mathcal{O}(11) \otimes f_*\omega_{X'}^5 \hookrightarrow f_*\omega_{X'}^6.$$

It is easy to see that ϕ_6 is birational for X by virtue of Kollar's technique. \square

Both Proposition 5.2 and Proposition 5.3 imply

COROLLARY 5.4. *Let X be a smooth projective 3-fold of general type. Then ϕ_9 is birational if $p_g(X) \geq 2$.*

Acknowledgment. This paper was begun while I was visiting the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy in 1999 and was finally fulfilled when I stayed as a visiting fellow at the Mathematisches Institut der Universität Göttingen, Germany in 2000. I would like to thank F. Catanese for fruitful discussions during the preparation of it. Special thanks are due to hospital faculty members of the Mathematisches Institut der Universität Göttingen.

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